International Journal of Mathematics and
Computer Applications Research (IJMCAR)
ISSN (P): 2249-6955; ISSN (E): 2249-8060
Vol. 9, Issue 2, Dec 2019, 13-28

ON BOI-ALGEBRAS

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ABSTRACT

In this paper, the notion of BOI-algebra is introduced which is a generalization of BCI-algebras. The concepts of G-part, p-radical, BOI-ideal, closed BUI-ideal and fuzzy BOI-ideal of BOI-algebra are introduced. The Cartesian product of fuzzy BOI-ideals is introduced and studied. Several theorems are stated and proved.

KEYWORDS: BOI-Algebra, BOI-Ideal & Fuzzy BOI-Ideal

Received: May 14, 2019; **Accepted:** Jun 04, 2019; **Published:** Jul 04, 2019; **Paper Id.:** IJMCARDEC20192

Original Article

1. INTRODUCTION

Imai and Iséki introduced two classes of abstract algebras: BCK-algebra and BCI algebra ([10], [11]). These algebras have been extensively studied since they introduced. In ([8], [9]) Hu and Li introduced a wide class of abstract algebras: BCH-algebra which is a generalization of the notion of BCK and BCI-algebras and studied a few properties of these algebras. In 2002, Neggers and Kim [14] introduced a new notion, called a B-algebras, and obtained several results. Jun, Roh, and Kim [12] introduced a notion of BH-algebra, which is a generalization of B/BCI/BCK-algebras and generalize some theorems from the theory of BCI-algebras. In 2007, Walendziak [16] introduced a new notion, called a BF-algebra, which a generalization of B-algebra. Moreover, C. B. Kim and H. S. Kim [13] introduced BG-algebra as a generalization of B-algebra. In [15] Neggers, Ahn and Kim introduced Q-algebra, and also in [1] Ahn and Kim introduced QS-algebras, which are a generalization of BCH/BCI/BCK-algebras. In [3], Bandaru introduced a new notion called BRK-algebra which is a generalization of BCK/BCI/BCH/Q/QS/BM-algebras. In 2015, O. R. El-Gendy ([5], [7]) introduced the fuzzification of BRK-ideal of BRK-algebra and investigated some related properties, see also [6]. In this paper, we introduce a new notion, called a BOI-algebra which is a generalization of BCI/BCH –algebras and related to several classes of algebras of interest such as Q//BG/BH/BF -algebras. The concepts of G-part, p-radical, BOI-ideal are introduce, we consider the fuzzification of BOI-ideal of BOI-algebra. We investigate some related properties. The homomorphic inverse image of a fuzzy BOI-ideal is studied well. Moreover, we introduced the Cartesian product of fuzzy BOI-ideals. Several theorems are stated and proved.

2. PRELIMINARIES

Definition 2.1 [10, 11]

A BCI-algebra $(Y;*,0)$ is a nonempty set Y with a constant 0 and a binary

operation "$*$" satisfies the following axioms:

(A$_1$) $((x*y)*(x*z))*(z*y)=0$,

(A$_2$) $(x*(x*y))*y=0$,

(A$_3$) $x*x=0$,

(A$_4$) $x*y=0$ and $y*x=0$ imply $x=y$, for all $x,y,z\in Y$.

A BCI-algebra is said to be a BCK-algebra if it satisfies

(A$_5$) $0*x=0$, for all $x\in Y$.

Definition 2.2 [8]

A BCH-algebra is an algebra $(Y;*,0)$ satisfying (A$_3$), (A$_4$) and (A$_6$): $(x*y)*z=(x*z)*y$.

Definition 2.3 [14]

A B-algebra is an algebra $(Y;*,0)$ satisfying (A$_3$), (A$_7$): $x*0=x$ and (A$_8$): $(x*y)*z=x*(z*(0*y))$.

A B-algebra is said to be 0-commutative if $a*(0*b)=b*(0*a)$ for any $a,b\in Y$.

Definition 2.4 [12]

A BH-algebra is an algebra $(Y;*,0)$ satisfying (A$_3$), (A$_4$) and (A$_7$).

Definition 2.5 [16]

A BF-algebra is an algebra $(Y;*,0)$ satisfying (A$_3$), (A$_7$) and (A$_9$): $0*(x*y)=(y*x)$.

Definition 2.6 [13]

A BG-algebra is an algebra $(Y;*,0)$ satisfying (A$_3$), (A$_7$) and (A$_{10}$): $(x*y)*(0*y)=x$.

Definition 2.7 [15]

A Q-algebra is an algebra $(Y;*,0)$ satisfying (A$_3$), (A$_6$) and (A$_7$). A Q-algebra is said to be a QS-algebra if it satisfies the additional axiom: (A$_{11}$) $(x*y)*(x*z)=(z*y)$.

Definition 2.8 [2]

A BP-algebra is an algebra $(Y;*,0)$ satisfying (A$_3$), (A$_{12}$): $x*(x*y)=y$ and (A$_{13}$): $(x*z)*(y*z)=(x*y)$.

3. BOI-ALGEBRAS

Definition 3.1

A BOI-algebra $(X;*,0)$ (i. e., a nonempty set X with a binary operation "$*$" and a constant 0) satisfying the following axioms:

(M$_1$) $x*x=0$,

(M_2) $x * (x * y) = y$,

(M_3) $(x * y) * z = (x * z) * y$, for all $x, y, z \in X$.

In *X*, we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

Example 3.2

Let $X = \{0,1,2\}$. Define $*$ on X as the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X;*,0)$ is a BOI-algebra.

Proposition 3.3

If $(X;*,0)$ is a BOI-algebra, then following conditions hold:

- $x * 0 = x$,
- $0 * (x * y) = y * x$,
- $(z * x) * (z * y) = y * x$
- $(x * z) * (y * z) = x * y$,
- $(x * y) * (0 * y) = x$,
- $x * (x * (x * y)) = x * y$,
- $(x * y) * x = 0 * y$,
- $x * (y * z) = (x * y) * (0 * z)$.

Proof

Let $(X;*,0)$ be a BOI-algebra and $x, y, z \in X$. Then

- Put $y = x$ in (M_2), we obtain $(x * (x * x)) = x * 0 = x$. (by M_1)
- $0 * (x * y) = (x * x) * (x * y) = (x * (x * y)) * x = y * x$, by ($M_1$), ($M_3$) and ($M_2$)
- $(z * x) * (z * y) = (z * (z * y)) * x = y * x$, by ($M_3$) and ($M_2$).

- $$\begin{aligned}(x*z)*(y*z) &= (0*(z*x))*(0*(z*y)) \\ &= (0*(0*(z*y)))*(z*x) \\ &= (z*y)*(z*x) = x*y \text{, by (2), } M_3 \text{ and } M_1.\end{aligned}$$

- $(x*y)*(0*y) = x*0 = x$, by (4) and (1).

- Straightforward. By (M_2).

- Straightforward. By (M_3) and (M_1).

- $x*(y*z) = (x*y)*((y*z)*y) = (x*y)*(0*z)$, by (4), (7).

The relation between BOI-algebra and other algebras are investigated and presented below.

Example 3.4

Let $X = \{0,1,2,3,4,5\}$. Define $*$ on X as the following table:

$*$	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then $(X;*,0)$ is a BOI-algebra.

Theorem 3.5

Every BOI-algebra is a BCI-algebra. But the converse needs not to be true in general.

Proof

Let $(X;*,0)$ be a BOI-algebra and $x, y, z \in X$.

Then, by Definition 3.1, $((x*y)*(x*z))*(z*y) = ((x*(x*z))*y)*(z*y) = (z*y)*(z*y) = 0$, and

$(x*(x*y))*y = y*y = 0$. Assume that $x*y = 0 = y*x$. Therefore

$x = y*(y*x) = y*0 = y*(y*y) = y$, and $y = x*(x*y) = x*0 = x*(x*x) = x$

Implies $x = y$. hence $(X;*,0)$ is BCI-algebra. For example, Example 3.4 is also BCI-algebra.

The converse of the theorem does not hold in general. To prove that considers the following example.

Example 3.6

Let $X = \{0, 1, 2\}$ in which $*$ is given by the table

$*$	0	1	2
0	0	0	0
1	1	0	1
2	2	2	0

Then $(X;*,0)$ is a BCI-algebra, but not a BOI-algebra, because $0*(0*1)=0*0=0\neq 1$.

Theorem 3.7

Every BOI-algebra is a BCH-algebra. But the converse needs not to be true.

Proof

Straight forward.

The converse of the theorem needs not to be true. To prove that considers the following example.

Example 3.8

Let $X=\{0,1,2,3\}$. Define $*$ on X as the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	3	3
2	2	0	0	2
3	3	0	0	0

Then $(X,*;0)$ is a BCH-algebra, but not a BOI-algebra, because the axiom (M_2) is not held, since $0*(0*3)=0*0=0\neq 3$.

Theorem 3.9

Every BOI-algebra is a Q-algebra.

Proof

Straight forward. The converse of the theorem needs not to be true. To prove that considers the following example.

Example 3.10

Let X = {0, 1, 2} in which $*$ is given by the following table:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then $(X;*,0)$ is both a Q-algebra and QS-algebra, but not a BOI-algebra, since (M_2) does not hold.

Theorem 3.11

If $(X;*,0)$ is a BOI-algebra, then it is a BH-algebra.

Proof

Straight forward.

Theorem 3.12

If $(X;*,0)$ is a BOI-algebra, then $(X;*,0)$ is a BF-algebra.

Proof

Straight forward.

Theorem 3.13

If $(X;*,0)$ is a BOI-algebra, then $(X;*,0)$ is a BG-algebra.

Proof

Straight forward.

The converse of Theorems 3.11, 3.12 and 3.13 need not be true. To prove that considers the following example.

Example 3.14

Let $X = \{0,1,2,3,4,5\}$. Define $*$ on X as the following table:

$*$	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X,*;0)$ is a BF/BG/BH-algebra which is not a BOI-algebra, since (M_2) does not hold.

Example 3.15

Let $X = \{0,1,2\}$. Define $*$ on X as the following table:

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0

Then $(X;*,0)$ is a BOI-algebra which is not BP-algebra.

It is easy to see BP-algebra and BOI-algebra are different notations. For example, Example 3.15 is a BOI-algebra which is not a BP-algebra. The axiom (A_{13}) is not held, since $(0*2)*(1*2) \neq (0*1)$.

4. BOI-IDEALS

In this section, we define sub algebra, ideal, closed ideal, BOI-ideal and homomorphism of BOI-algebra.

Definition 4.1

A nonempty subset I of a BOI-algebra X is called a sub algebra of X if $x * y \in I$ for all $x, y \in$ I

Definition 4.2

A non-empty subset I of a BOI-algebra$(X;*,0)$is called an ideal if for any$x, y \in X$:

I_1) $0 \in I$,

I_2) $x * y \in I$ *and* $y \in I$ *imply* $x \in I$.

Definition 4.3

A non-empty subset M of a BOI-algebra$(X;*,0)$is called a BOI-ideal if for any$x, y, z \in X$:

B_1) $0 \in M$,

B_2)$x * y \in M$ *and* $(x * z) * y \in M$ *imply* $z \in M$.

Obviously, $\{0\}$ and X are ideals of a BOI-algebra X. we call $\{0\}$ and X the zero ideal and the trivial ideal of X, respectively. An ideal M is said to be proper if $M \neq X$.

Definition 4.4

ABOI-ideal M of a BOI-algebra X is called a closed ideal of X if $0 * x \in M$ for all $x \in M$.

Example 4.5

Let $X = \{0,1,2,3\}$ Define $*$ on X as the following table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X,*;0)$ is a BOI-algebra, $\{0,1\}$ is a sub algebra, an ideal, BOI-ideal of X and a closed ideal.

Theorem 4.6

Every closed BOI-ideal of BOI-algebra is a sub algebra.

Proof

Let M be a closed BOI-ideal of a BOI-algebra$(X;*,0)$ and $x, y \in M$. Then

$0 * y \in M$. By Definition 3.1, $(x * y) * x = 0 * y \in M$. Since $M$is a BOI-ideal and $x \in M$, we have $x * y \in M$. Then M is a subalgebra of X.

Definition 4.7

(Homomorphism of BOI-algebra). Let $(X,*,0)$ and $(Y,*',0')$ be BOI-algebra. A mapping $f: X \to Y$ is said to be a homomorphism if $f(x * y) = f(x) *' f(y)$, for all $x, y \in X$.

Proposition 4.8

Let $(X,*,0)$ and $(Y,*',0')$ be BOI-algebra, and the mapping $f: X \to Y$ is a homomorphism of BOI-algebra, then the $\ker(f)$ is a BOI-ideal.

Proof

Let $(x * z) * y \in kerf$, $x * y \in kerf$. Then $f\big((x * z) * y\big) = 0', f(x * y) = 0'$. By Definition 3.1, $0' = f((x * y) * \big((x * z) * y\big)) = f(x * y) *' f\big((x * z) * y\big) = f(x * y) *' (f(x * y) * 'f(z)) = 0' *' \big(0' *' f(z)\big) = f(z)$. Hence $z \in kerf$.

Theorem 4.9

Let $(X;*,0)$ and $(Y;*',0')$ be BOI-algebra and let M be a BOI-ideal of Y. If $f: X \to Y$ is onto homomorphism, then $f^{-1}(M)$ is a BOI-ideal of X.

Proof

Let M be a BOI-ideal of Y, and $f: X \longrightarrow Y$is onto homomorphism.

Therefore $0' \in M \implies 0 = f^{-1}(0') \in f^{-1}(M)$.

Assume that$(x * z) * y \in f^{-1}(M), and\ x * y \in f^{-1}(M)$.

Then $f\big((x * z) * y\big) = (f(x) *' f(z)) * 'f(y) \in M$, and $f(x * y) = f(x) *' f(y) \in M$. Since M is a BOI-ideal of Y, then $f(z) \in M$, so that $z \in f^{-1}(M)$. Hence $f^{-1}(M)$is a BOI-ideal.

5. G-PART OF BOI-ALGEBRAS

In this section, we define G-part, $p - radical$of a BOI-algebra, and investigate the properties of the G-part in BOI-algebra.

Lemma5.1

If $(X;*,0)$ is a BOI-algebra and $x * y = x * z$ for $x, y, z \in X$, then $0 * y = 0 * z$.

Proof

Let $(X;*,0)$ be a BOI-algebra and $x, y, z \in X$. Then by (M_1) and (M_3),

$x * y = x * z \Rightarrow (x * y) * x = (x * z) * x \Rightarrow (x * x) * y = (x * x) * z \Rightarrow 0 * y = 0 * z$.

Definition 5.2

Let $(X;*,0)$ be a BOI-algebra. For any nonempty subset S of X, we define

$G(S) = \{x \in S | 0 * x = x\}$.

In particular, If $S = X$ then we say that $G(X)$is the G-part of X.

Theorem 5.3

Let $(X;*,0)$ be a BOI-algebra. Then a left cancelation law holds in $G(X)$.

Proof

Let $x, y, z \in G(X)$ with $x * y = x * z$. Then, by Lemma 5.1, $0 * y = 0 * z$. Since $y, z \in G(X)$, we obtain $y = z$.

Proposition 5.4

Let $(X;*,0)$ be a BOI-algebra. Then $x \in G(X)$if and only if $0 * x \in G(X)$.

Proof

If $x \in G(X)$, then $0 * x = x \Rightarrow 0 * (0 * x) = 0 * x$. Hence $0 * x \in G(X)$. Conversely, if $0 * x \in G(X)$, then $0 * (0 * x) = 0 * x$. By appling Theorem 5.3, we obtain $0 * x = x$. Therefore $x \in G(X)$.

Definition 5.5

For any BOI-algebra$X$, the set

$B(X) = \{x \in X | 0 * x = 0\}$

Is called the $p - radical$ of X. Clearly $B(X)$is a sub algebra.

Definition 5.6

A BOI-algebra X is said to be $p - semisimple$ if $B(A) = \{0\}$.

Clearly, for any BOI-algebra$X$, $G(X) \cap B(X) = \{0\}$.

Proposition 5.7

If $(X;*,0)$ is a BOI-algebra and $x, y \in X$, then $y \in B(X) \Leftrightarrow (x * y) * x = 0$.

Proof

Let $(X;*,0)$ be a BOI-algebra and $x, y \in X$. Then, by (M_1) and (M_3),

$y \in B(X) \Leftrightarrow 0 * y = 0 \Leftrightarrow (x * y) * x = (x * x) * y = 0 * y = 0$.

Proposition 5.8

Let $(X;*,0)$ be a BOI-algebra. Then $B(X)$ is a BOI-ideal of X.

Proof

By Definition 3.1 and Proposition 5.7. Since $(0 * 0) * 0 = 0$, then $0 \in B(X)$.

Let $x*y \in B(X)$ and $(x*z)*y \in B(X)$. If $((x*y)*((x*z)*y)) \in B(X) \implies$

$0*\left((x*y)*\left((x*z)*y\right)\right)=0$. Then$0*\left((x*y)*\left((x*z)*y\right)\right)=0*\left((x*y)*\left((x*y)*z\right)\right)=0*z=0$

Hence $z \in B(X)$. Therefore $B(X)$ is a BOI-ideal of X.

Proposition 5.9

Let $(X;*,0)$ be a BOI-algebra. Then $B(X)$ is an ideal of X.

Proof

Since $(0*0)*0=0$, by Proposition 5.7, $0 \in B(X)$. Let $x*y \in B(X)$ and $y \in B(X)$. Then, by (M_3) and Proposition 5.7, $((x*y)*x)*(x*y)=0$. Then $\left((x*y)*(x*y)\right)*x=0*x=0$. Hence $x \in B(X)$. Therefore $B(X)$ is an ideal of X.

Proposition 5.10

If S is a subalgebra of a BOI-algebra $(X;*,0)$, then $G(X) \cap S = G(S)$.

Proof

Clearly, $G(X) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0*x=x$ and $x \in S \subseteq X$. Hence, $x \in G(X)$. Therefore, $x \in G(X) \cap S$. Thus, $G(X) \cap S = G(S)$.

Theorem 5.11

Let $(X;*,0)$ be a BOI-algebra. If $G(X)=X$, then X is a $p-semisimple$.

Proof

Assume that $G(X)=X$. Then $\{0\}=G(X) \cap B(X)=X \cap B(X)=B(X)$. Hence, X is $p-semisimple$.

Theorem 5.12

Let $(X;*,0)$ be a BOI-algebra and let M be a BOI-ideal of X. Then M contains the $G-partG(X)$ of X.

Proof

If $x \in G(X)$, then $(0*x)*x=x*x=0 \in M$, and $0*x=x \in G(X)$. Since M is a BOI-ideal of X, therefore $x=0*x \in M$. Hence $G(X) \subseteq M$.

6. FUZZY BOI-IDEAL

In this section, we define the fuzzy BOI-ideal of a BOI-algebra and investigate the properties of fuzzy BOI-ideal of BOI-algebra.

Definition 6.1

Let $(X;*,0)$ be a BOI-algebra. A fuzzy set μ in X is called a fuzzy BOI-ideal of X if it satisfies:

F_1)$\mu(0) \geq \mu(x)$,

F_2) $\mu(z) \geq \min\{\mu(x*y), \mu\left((x*z)*y\right)\}, for\ all\ x,y,z \in X$.

Lemma 6.2

If μ is a fuzzy BOI-ideal of BOI-algebra *X*, then $\mu(0 * x) = \mu(x)$ for all $x \in X$.

Proof

Let μ be a fuzzy BOI-ideal of X. By Definition 3.1,

$\mu(0 * x) = \min\left\{\mu(x * y), \mu\left(\left(x * (0 * x)\right) * y\right)\right\} = \min\{\mu(0), \mu((x * y) * (0 * x))\} = \min\{\mu(0), \mu(0 * (0 * x))\} = \min\{\mu(0), \mu(x)\} = \mu(x)$.

Lemma 6.3

If μ is a fuzzy BOI-ideal of BOI-algebra *X*, then$x \leq y implies\ \mu(x) \geq \mu(y) for\ all\ x, y, z \in X$.

Proof

Let μ be a fuzzy BOI-ideal of X. By Definition 3.1 and Lemma 4.2, if $x \leq y \Longrightarrow x * y = 0$.

Then, $\mu(x) \geq \min\{\mu(x * y), \mu((x * x) * y)\} = \min\{\mu(0), \mu(0 * y)\} = \mu(0 * y) = \mu(y)$.

Theorem 6.4

The intersection of any set of fuzzy BOI-ideals of BOI-algebra is also a fuzzy BOI-ideal.

Proof

Let { μ_i } be a family of fuzzy BOI-ideals of BOI-algebras *X*. Then for any $x, y, z \in X$,

$(\bigcap \mu_i)(0) = \inf(\mu_i(0)) \geq \inf(\mu_i(x)) = (\bigcap \mu_i)(x)$, and

$$\begin{aligned}(\bigcap \mu_i)(z) = \inf(\mu_i(z)) &\geq \inf\ (\min\{\mu_i(x * y), \mu_i((x * z) * y)\}) \\ &= \min\{\inf(\mu_i(x * y)), \inf(\mu_i((x * z) * y))\} \\ &= \min\{(\bigcap \mu_i)(x * y), (\bigcap \mu_i)(\mu_i((x * z) * y))\}\ .\end{aligned}$$

Definition 6.5

Let *f* be a mapping from the set *X* to the set *Y*. If μ is a fuzzy subset of *X*,

then the fuzzy subset *B* of *Y* defined by

$$\mu\ f^{-1}(y) = B(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x), \text{ if } f^{-1}(y) = \{x \in X, f(x) = y\} \neq \phi \\ 0 \qquad\qquad\qquad\qquad \text{otherwise} \end{cases},$$

for all $y \in Y$ is called the image of μ under *f*.

Similarly, if *B* is a fuzzy subset of *Y*, then the fuzzy subset defined by $B(f(x)) = \mu(x)$ for all $x \in X$, is said to be the inverse image of *B* under *f*.

Theorem 6.6

An onto homomorphic inverse image of a fuzzy BOI-ideal is also a fuzzy BOI-ideal.

Proof

Let $f : X \to X'$ be an onto homomorphism of BOI-algebras, and β is a fuzzy BOI-ideal of X' and μ is the inverse image of β under *f*. Then by, Definition (6.5), $\beta(f(x)) = \mu(x)$, for all $x \in X$, $\mu(0) = \beta(f(0)) \geq \beta(f(x)) = \mu(x)$. Let $x, y \in X$, then

$\mu(z) = \beta(f(z)) \geq \min\{\beta((f(x) *' f(y)), \beta((f(x) *' f(z)) *' f(y))\} = \min\{\beta(f(x * y), \beta(f((x * z) * y))\}$ $= \min\{\mu(x * y), \mu((x * z) * y))\}$. Hence $\mu(z) = \beta(f(z)) = (\beta \circ f)(z)$ is a fuzzy BOI-ideal of *X*.

7. CARTESIAN PRODUCT OF FUZZY BOI-IDEAL

Definition 7.1[4]

A fuzzy relation on any set S is a fuzzy subset $\mu : S \times S \to [0,1]$.

Definition 7.2[4]

If μ is a fuzzy relation on a set S and β is a fuzzy subset of S, then μ is a fuzzy relation on β if $\mu(x, y) \leq \min\{\beta(x), \beta(y)\}$ for all $x, y \in S$.

Definition 7.3[4]

Let μ and β be fuzzy subsets of a set S. The Cartesian product of μ and β is defined by $(\mu \times \beta)(x, y) = \min\{\mu(x), \beta(y)\}$ for all $x, y \in S$.

Definition 7.4[4]

If β is a fuzzy subset of a set S, the strongest fuzzy relation on S that is a fuzzy relation on β is μ_β given by $\mu_\beta(x, y) = \min\{\beta(x), \beta(y)\}$ for all $x, y \in S$.

Corollary 7.5

Let $(X; *, 0)$ and $(Y; *, 0')$ be BOI-algebras, we define $*$ on $X \times Y$ by For every $(x_1, x_2), (y_1, y_2) \in X \times Y$ $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$ then $(X \times Y; *, (0,0'))$ is a BOI-algebra.

Proof

Let $(X; *, 0)$ and $(Y; *, 0')$ be BOI-algebras. For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times Y$, then

1. $(x_1, x_2) * (x_1, x_2) = (x_1 * x_1, x_2 * x_2) = (0,0')$

2. $(x_1, x_2) * ((x_1, x_2) * (y_1, y_2)) = (x_1, x_2) * (x_1 * y_1, x_2 * y_2) = (x_1 * (x_1 * y_1), x_2 * (x_2 * y_2)) = (y_1, y_2)$.

3 $\left((x_1,x_2)*(y_1,y_2)\right)*(z_1,z_2)=\left((x_1*y_1)*z_1,(x_2*y_2)*z_2\right)=\left((x_1*z_1)*y_1,(x_2*z_2)*y_2\right)$

$=\left((x_1,x_2)*(z_1,z_2)\right)*(y_1,y_2)$.

Then $(X\times Y\,;*,(0,0'))$ is a BOI-algebra.

Proposition 7.6

For a given fuzzy subset β of a BOI-algebra X, let μ_β be the strongest fuzzy relation on X. If μ_β is a fuzzy BOI-ideal of $(X\times X;*,(0,0))$, then $\beta(0)\geq\beta(x)$ for all $x\in X$.

Proof

Since μ_β is a fuzzy BOI-ideal of $X\times X$. Therefore $\mu_\beta(0,0)\geq\mu_\beta(x,x)$. So that $\mu_\beta(0,0)=\min\{\beta(0),\beta(0)\}\geq\mu_\beta(x,x)=\min\{\beta(x),\beta(x)\}$.

This implies that $\beta(0)\geq\beta(x)$

Theorem 7.7

Let $(X\,;*,0)$ and $(Y;*,0')$ be BOI-algebras. μ and β be fuzzy BOI-ideals of *X* and *Y* respectively. Then $\mu\times\beta$ is a fuzzy BOI-ideal of $(X\times Y\,;*,(0,0'))$.

Proof

Assume that μ and β be a fuzzy BOI-ideal *X* and *Y*.

Then for all $(x,y)\in X\times Y$

$$(\mu\times\beta)(0,0')=\min\{\mu(0),\beta(0')\}\geq\min\{\mu(x),\beta(y)\}=(\mu\times\beta)(x,y).$$

Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times Y$

$\min\left\{(\mu\times\beta)\left((x_1,x_2)*(y_1,y_2)\right),(\mu\times\beta)\left(\left((x_1,x_2)*(z_1,z_2)\right)*(y_1,y_2)\right)\right\}=\min\{(\mu\times\beta)(x_1*y_1,x_2*y_2),(\mu\times\beta)\left((x_1*z_1)*y_1,(x_2*z_2)*y_2\right)\}=\min[\min\{\mu(x_1*y_1),\beta(x_2*y_2)\},\min\{\mu\,(x_1*z_1)*y_1),\beta((x_2*z_2)*y_2)]=\min[\min\{\mu(x_1*y_1),\mu((x_1*z_1)*y_1)\},\min\{\beta(x_2*y_2),\beta((x_2*z_2)*y_2)]\leq\min\{\mu(z_1),\beta(z_2)\}=(\mu\times\beta)(z_1,z_2)$.

Theorem 7.8

If β is a fuzzy subset of BOI-algebras *X*, and μ_β is the strongest fuzzy relation on *X* then, β is a fuzzy BOI-ideal of *X* if μ_β is a fuzzy BOI-ideal of $(X\times X;*,(0,0'))$.

Proof

Suppose that, β is a fuzzy subset of a BOI-algebra$X$ and μ_β is the strongest fuzzy relation on *X*. Then $\mu_\beta(0,0') = \min\{\beta(0), \beta(0')\} \geq \min\{\beta(x), \beta(y)\} = \mu_\beta(x,y)$, for all $(x,y) \in X \times X$.

For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we get that

$$\begin{aligned}
\mu_\beta(z_1, z_2) &= \min\{\beta(z_1), \beta(z_2)\} \\
&\geq \min[\min\{\beta(x_1 * y_1), \beta((x_1 * z_1) * y_1)\}, \min\{\beta(x_2 * y_2), \beta((x_2 * z_2) * y_2)\}] \\
&= \min[\min\{\beta(x_1 * y_1), \beta(x_2 * y_2)\}, \min\{\beta((x_1 * z_1) * y_1), \beta((x_2 * z_2) * y_2)\}] \\
&= \min[\mu_\beta((x_1 * y_1), (x_2 * y_2)), \mu_\beta((x_1 * z_1) * y_1, (x_2 * z_2) * y_2)] \\
&= \min\{\mu_\beta((x_1, x_2) * (y_1, y_2)), \mu_\beta(((x_1, x_2) * (z_1, z_2)) * (y_1, y_2))\} .
\end{aligned}$$

Hence μ_β is a fuzzy BOI-ideal of $X \times X$.

8. CONCLUSIONS AND FUTURE RESEARCH

In this paper, we have introduced the concept of BOI-algebra and studied their properties. in addition, we have defined $G-part$, $p-radical$, BOI-ideal and fuzzy BOI-ideal of BOI-algebra. Finally, we studied and proved the Cartesian product of fuzzy BOI-ideal.

In the future, these definitions and fundamental results can be applied to some different algebraic structures. There are more topics that could take advantage of BOI-ideal. Like for example cubic intuitionistic BOI-ideal of BOI-algebra, cubic fuzzy BOI-ideal of BOI-algebra, and cubic soft BOI-ideal in BOI-algebra. There are many other aspects which should be explored and studied in the area of BOI-algebra such as anti-fuzzy BOI-ideal of BOI-algebra, interval-valued fuzzy BOI-ideal of BOI-algebra, intuitionistic fuzzy BOI-ideal of BOI-algebra, fuzzy soft BOI-ideal of BOI-algebra, bipolar fuzzy BOI- ideal of BOI-algebra, doubt intuitionistic fuzzy BOI-ideal of BOI-algebra, fuzzy derivations BOI-ideal of BOI-algebra, and interval-valued intuitionistic fuzzy BOI-ideal of BOI-algebra. It is our hope that this work would other foundations for further study of the theory of BOI-algebra.

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